A Sequence of Real Numbers Converging to Zero But Not in Any Of the l_p Spaces $(1 \leq p < \infty)$

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Abstract

This small note is for giving a rigorous solution to Exercise 1.2-4 of *Introductory Functional Analysis and Applications* by Erwin Otto Kreyszig.

Question

Find a sequence which converges to 0, nut is not in any space l_p, where $1 \leq p < \infty$.

Solution

Let $x : \mathbb{N} \longrightarrow \mathbb{R}$ be a sequence of real numbers such that for every $n \in \mathbb{N}$, we have

$$x_{2^n-1} = \frac{1}{n};$$

$$x_{2^n-1} = x_{2^n} = x_{2^n+1} = \cdots = x_{2(2^n-1)}.$$

We note that x is a well-defined sequence as the second part of the definition of x explicitly defines x_1 and assigns a value to every term between x_{2^k-1} and $x_{2^{k+1}-1}$ for every $k \in \mathbb{N}$. The second condition ensures that there are a total of $2(2^n-1)-(2^n-2) = 2^n$ terms in the sequence whose value is $x_{2^n-1} = \dfrac{1}{n}$, for every choice of $n \in \mathbb{N}$. This shall be useful for showing that $x \notin l_p$ for every $p \geq 1$. Now, we attempt to show that $\lim\limits_{n\to\infty} x_n = 0$. For this, we use the Monotone Convergence Theorem for real sequences. Let $k, m \in \mathbb{N}$ be such that $k < m$. Then, we have two cases. The first one is that $k = 2^{n_0} - 1$ for some $n_0 \in \mathbb{N}$. Then, $x_k = \dfrac{1}{n_0}$. If $m < 2^{n_0+1} - 1$, then $x_k = x_m$ otherwise $x_k > x_m$. The second case is that $k \neq 2^n - 1$ for every $n \in \mathbb{N}$. In this case, we claim that $2^{n_0} - 1 < k < 2^{n_0+1} - 1$, for some $n_0 \in \mathbb{N}$. If this doesn't hold, then we get the boundedness of $\mathbb{N}$, which is not possible. Thus, proceeding as before, we can show that $x_k \geq x_m$. Thus, $(x_k)_{k=1}^{\infty}$ is a decreasing sequence. Now, we show that

$$\{x_k : k \in \mathbb{N}\} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Let $\alpha \in \{x_k : k \in \mathbb{N}\}$. Then $\alpha = x_z$ for some $z \in \mathbb{N}$, and by definition of x, we have $x_z = \dfrac{1}{z_0}$ for some $z_0 \in \mathbb{N}$. Thus, $\alpha \in \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$. Conversely, let $\beta \in \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$. Then, $\beta = \dfrac{1}{v}$ for

some $v \in \mathbb{N}$. So, $\beta \in \{x_k : k \in \mathbb{N}\}$ as $x_{2^v-1} = \dfrac{1}{v} = \beta$. Thus, we obtain the desired inequality. From Real Analysis, we have

$$\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \inf\{x_k : k \in \mathbb{N}\} = 0.$$

Thus, Monotone Convergence Theorem makes us conclude that $\lim\limits_{n\to\infty} x_n = 0$. Now, to show that $(x_n)_{n=1}^{\infty} \notin l_p$ for every $p \geq 1$, we assume the contrary. So let $(x_n)_{n=1}^{\infty} \in l_P$ for some $P \geq 1$. Therefore

$$\sum_{n=1}^{\infty} |x_n|^P = \sum_{n=1}^{\infty} x_n^P = S < \infty.$$

Let $(S_n)_{n=1}^{\infty}$ be the sequence of partial sums of $(x_n^P)_{n=1}^{\infty}$. Then, $\lim\limits_{n\to\infty} S_n = S$ and so, every sub-sequence of $(S_n)_{n=1}^{\infty}$ converges to S. Define $h : \mathbb{N} \longrightarrow \mathbb{N}$ as $h(n) = 2(2^n - 1)$ for all $n \in \mathbb{N}$. Then, h is a strictly increasing function and so, the composition $s_\circ h$ is a sub-sequence of the sequence $s = (S_n)_{n=1}^{\infty}$. Now let $(y_n)_{n=1}^{\infty} = \left(S_{h(n)}\right)_{n=1}^{\infty}$. Then, we see that

$$y_1 = \sum_{i=1}^{h(1)} x_i^P = x_1 + x_2 = 1 + 1 = \sum_{i=1}^{1} 2^i \frac{1}{i^P}.$$

On a similar note, using the second condition in the definition of x, we have

$$y_2 = \sum_{i=1}^{h(2)=6} x_i^P = (x_1 + x_2) + (x_3 + x_4 + x_5 + x_6) = (1+1) + 4 \times \frac{1}{2^P} = \sum_{i=1}^{2} 2^i \frac{1}{i^P}.$$

Now, as the induction hypothesis, let

$$y_k = \sum_{i=1}^{h(k)} x_i^P = \sum_{i=1}^{k} 2^i \frac{1}{i^P}.$$

To prove the inductive step, we have

$$\begin{aligned}
y_{k+1} &= \sum_{i=1}^{h(k+1)} x_i^P \\
&= \sum_{i=1}^{h(k)} x_i^P + \sum_{i=h(k)+1}^{h(k+1)} x_i^P \\
&= \sum_{i=1}^{k} 2^i \frac{1}{i^P} + \sum_{i=h(k)+1}^{h(k+1)} x_i^P \\
&= \sum_{i=1}^{k} 2^i \frac{1}{i^P} + \sum_{i=2^{k+1}-1}^{2(2^{k+1}-1)} x_i^P \\
&= \sum_{i=1}^{k} 2^i \frac{1}{i^P} + \underbrace{\frac{1}{k+1} + \frac{1}{k+1} + \frac{1}{k+1} + \cdots + \frac{1}{k+1}}_{2^{k+1} \text{ times}}
\end{aligned}$$

So, we have $y_{k+1} = \displaystyle\sum_{i=1}^{k+1} 2^i \frac{1}{i^P}$ and hence $y_n = \displaystyle\sum_{i=1}^{n} 2^i \frac{1}{i^P}$ for all $n \in \mathbb{N}$. Thus $\left(\displaystyle\sum_{i=1}^{n} 2^i \frac{1}{i^P}\right)_{n=1}^{\infty}$ is a sub-sequence of $(S_n)_{n=1}^{\infty}$ and so, it follows that

$$\sum_{i=1}^{\infty} 2^i \frac{1}{i^P} = S < \infty.$$

But, we have

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^P} \times \frac{n^P}{2^n} = 2 \lim_{n\to\infty} \left(\frac{n}{1+n}\right)^p = 2 \times 1 > 1.$$

Thus, from Ratio Test of Convergence, $\sum_{i=1}^{\infty} 2^i \frac{1}{i^P}$ is divergent. So, we arrive at a contradiction since an infinite series is both convergent and divergent. Therefore, $x \notin l_p$ for any $p \geq 1$.